

A note on $n!$ modulo p

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Abstract

Let p be a prime, $\varepsilon > 0$ and $0 < L + 1 < L + N < p$. We prove that if $p^{1/2+\varepsilon} < N < p^{1-\varepsilon}$, then

$$\#\{n! \pmod{p}; L + 1 \leq n \leq L + N\} > c(N \log N)^{1/2}, \quad c = c(\varepsilon) > 0.$$

We use this bound to show that any $\lambda \not\equiv 0 \pmod{p}$ can be represented in the form $\lambda \equiv n_1! \dots n_7! \pmod{p}$, where $n_i = o(p^{11/12})$. This slightly refines the previously known range for n_i .

1 Introduction

In what follows, p is a large prime number. For integers L and N with

$$0 < L + 1 < L + N < p$$

we consider the set

$$\mathcal{A}(L, N) = \left\{ n! \pmod{p}; L + 1 \leq n \leq L + N \right\}.$$

From the observation

$$\{1\} \cup \{L + 2, \dots, L + N\} \pmod{p} \subset \frac{\mathcal{A}(L, N)}{\mathcal{A}(L, N)} \quad (1)$$

it follows that

$$|\mathcal{A}(L, N)| \geq N^{1/2}.$$

In particular, we trivially have $|\mathcal{A}(0, p - 1)| \geq (p - 1)^{1/2}$. From the result of García [7] on the cardinality of product of two factorials modulo p it follows that $|\mathcal{A}(0, p - 1)| > cp^{1/2}$ for any constant $c < \sqrt{\frac{41}{24}}$ and sufficiently

large prime p . The conjecture is that $|\mathcal{A}(0, p)|$ asymptotically behaves like $(1 - e^{-1})p$, see [5] and [9].

Improving on the trivial bound, Klurman and Munsch [10] proved the bound

$$|\mathcal{A}(L, N)| \geq cN^{1/2} \quad (2)$$

with $c = \sqrt{\frac{3}{2}}$ and $p^{1/4+\varepsilon} < N < p$. We remark that the condition $N > p^{1/4+\varepsilon}$ can be relaxed if one combines (1) with [4, Theorem 2] (see, also, the work [3] on congruences with variables from short intervals).

In the present note, using a consequence of Bombieri's bound on exponential sums over algebraic curves, we show that if $p^{1/2+\varepsilon} < N = o(p)$, then the constant c in (2) can be taken arbitrarily large. We then apply this result to the problem of representability of residue classes as a product of seven factorials with small variables.

Theorem 1. *Let $p^{1/2+\varepsilon} < N < 0.1p$. Then*

$$\left| \frac{\mathcal{A}(L, N)}{\mathcal{A}(L, N)} \right| > c_0 N \log(p/N)$$

for some $c_0 = c_0(\varepsilon) > 0$.

From Theorem 1 it follows, in particular, that for $p^{1/2+\varepsilon} < N < 0.1p$ we have the bound

$$|\mathcal{A}(L, N)| > c_0 (N \log(p/N))^{1/2}$$

for some $c_0 = c_0(\varepsilon) > 0$.

Garaev, Luca and Shparlinski [6] proved that any $\lambda \not\equiv 0 \pmod{p}$ can be represented in the form

$$\prod_{i=1}^7 n_i! \equiv \lambda \pmod{p},$$

where $n_i \ll p^{11/12+\varepsilon}$. Garcia [8] improved this condition to $n_i \ll p^{11/12}$. Using Theorem 1 we can slightly improve this as follows.

Theorem 2. *Any $\lambda \not\equiv 0 \pmod{p}$ can be represented in the form*

$$\prod_{i=1}^7 n_i! \equiv \lambda \pmod{p},$$

where the positive integers n_1, \dots, n_7 satisfy

$$\max\{n_i | i = 1, \dots, 7\} \ll \frac{p^{11/12}}{(\log p)^{1/2}}.$$

2 Lemmas

We need the following special case of the results of Bombieri [1, Theorem 6] and Chalk and Smith [2, Theorem 2]. As usual, \mathbb{F}_p denotes the field of residue classes modulo p .

Lemma 1. *Let $(b_1, b_2) \in \mathbb{F}_p \times \mathbb{F}_p$ be nonzero and $f(x, y) \in \mathbb{F}_p[x, y]$ be a polynomial of degree $d \geq 1$ with the following property: there is no $c \in \mathbb{F}_p$ for which the polynomial $f(x, y)$ is divisible by $b_1x + b_2y + c$. Then*

$$\left| \sum_{f(x,y)=0} e^{2\pi i(b_1x+b_2y)/p} \right| \leq 2d^2p^{1/2}.$$

We remark that the factor 2 on the right hand side can be removed, but it is not essential in our application.

The following lemma is due to Ruzsa. It will be used in the proof of Theorem 2.

Lemma 2. *For any finite subsets X, Y, Z of an abelian group we have*

$$|X - Y| \leq \frac{|X + Z||Z + Y|}{|Z|}.$$

In the proof of Theorem 2 we will also need the following estimate of character sums with factorials from the work of García [8].

Lemma 3. *For any positive integer N the following bound holds:*

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq N} \sum_{m \leq N} \chi((n+m)!) \right| \ll N^{7/4} p^{1/8}.$$

3 Proof of Theorem 1

We can assume that p/N is sufficiently large in terms of ε . Let

$$M = \lfloor \min\{p^{0.1\varepsilon}, (p/N)^{0.1}\} \rfloor$$

For a positive integer $j \leq M$ we define the set

$$X_j = \left\{ \prod_{i=1}^j (x + L + i) \pmod{p}; \quad 1 \leq x < 0.6N \right\}.$$

Since the polynomial $\prod_{i=1}^j (x + L + i)$ has degree j , we have that

$$|X_j| \geq \frac{N}{2^j}. \quad (3)$$

Let us prove that for any $j \geq 2$ the following bound holds:

$$\#\{X_j \setminus (X_1 \cup \dots \cup X_{j-1})\} \geq \frac{N}{3^j}.$$

Note that

$$\begin{aligned} & \#\{X_j \setminus (X_1 \cup \dots \cup X_{j-1})\} \\ &= \#\{X_j \setminus ((X_j \cap X_1) \cup \dots \cup (X_j \cap X_{j-1}))\} \\ &\geq |X_j| - |X_j \cap X_1| - \dots - |X_j \cap X_{j-1}|. \end{aligned}$$

Therefore, in view of (3) we get

$$\#\{X_j \setminus (X_1 \cup \dots \cup X_{j-1})\} \geq \frac{N}{2^j} - |X_j \cap X_1| - \dots - |X_j \cap X_{j-1}|. \quad (4)$$

We shall obtain upper bound for the cardinality $|X_j \cap X_k|$ for $1 \leq k \leq j-1$. Let $J(j, k)$ be the number of solutions of the congruence

$$\prod_{i=1}^j (x + L + i) \equiv \prod_{i=1}^k (y + L + i) \pmod{p}, \quad 1 \leq x, y < 0.6N.$$

Clearly,

$$|X_j \cap X_k| \leq J(j, k). \quad (5)$$

Denote

$$f(x, y) = \prod_{i=1}^j (x + L + i) - \prod_{i=1}^k (y + L + i) \in \mathbb{F}_p[x, y].$$

Following standard arguments, we write $J(j, k)$ in the form

$$\begin{aligned} J(j, k) &= \sum_{\substack{x \leq 0.6N, y \leq 0.6N \\ f(x, y) = 0}} 1 \\ &\geq \frac{1}{p^2} \sum_{b_1=0}^{p-1} \sum_{b_2=0}^{p-1} \sum_{u < 0.6N} \sum_{v < 0.6N} \sum_{f(x, y) = 0} e^{2\pi i (b_1(x-u) + b_2(y-v))/p}. \end{aligned}$$

From the trivial bound we have that the number of solutions of the equation

$$f(x, y) = 0, \quad (x, y) \in \mathbb{F}_p \times \mathbb{F}_p$$

is not greater, than jp . We also recall the elementary estimates

$$\sum_{b=0}^{p-1} \left| \sum_{z < 0.6N} e^{2\pi i b z / p} \right| < p \log p.$$

Thus, separating the term that corresponds to $b_1 = b_2 = 0$, we obtain

$$J(j, k) \leq \frac{jN^2}{p} + (\log p)^2 \max_{(b_1, b_2)} \left| \sum_{f(x, y) = 0} e^{2\pi i (b_1 x + b_2 y) / p} \right|,$$

where the maximum is taken over the integers $0 \leq b_1, b_2 \leq p-1$ such that $(b_1, b_2) \neq (0, 0)$. Since $j > k \geq 1$, for any $a_1, a_2, a_3 \in \mathbb{F}_p$ the polynomials $f(X, a_1 X + a_2)$ and $f(a_3, X)$ are polynomials of degrees j and k in $\mathbb{F}_p[X]$. Therefore, $f(x, y)$ is not divisible by $b_1 x + b_2 y + c$ in $\mathbb{F}_p[x, y]$ and thus satisfies the condition of Lemma 1. Hence, applying Lemma 1 and taking into account that $j \leq M$, we get

$$J(j, k) \leq \frac{jN^2}{p} + O((\log p)^2 j^2 p^{1/2}) \leq \frac{N}{6j^2}.$$

This bound and (5) together with (4) implies that

$$\#\{X_j \setminus (X_1 \cup \dots \cup X_{j-1})\} \geq \frac{N}{2j} - \frac{(j-1)N}{6j^2} \geq \frac{N}{3j}.$$

Now we observe that

$$X_j \pmod{p} \subset \frac{\mathcal{A}(L, N)}{\mathcal{A}(L, N)}, \quad j = 1, 2, \dots, M.$$

Hence

$$\begin{aligned} \left| \frac{\mathcal{A}(L, N)}{\mathcal{A}(L, N)} \right| &\geq \#\{X_1 \cup X_2 \cup \dots \cup X_m\} \\ &= |X_1| + \sum_{j=2}^m \#\{X_j \setminus (X_1 \cup \dots \cup X_{j-1})\} \\ &\geq \sum_{j=1}^M \frac{N}{3j} \gg N \log M \gg N \log(p/N) \end{aligned}$$

and the result follows.

4 Proof of Theorem 2

Let $p^{0.51} < N < p^{0.99}$. For the brevity, denote $\mathcal{A} = \mathcal{A}(0, N)$. By Theorem 1 we have

$$\left| \frac{\mathcal{A}}{\mathcal{A}} \right| \gg N \log p, \quad |\mathcal{A}| \gg (N \log p)^{1/2}.$$

Application of Lemma 2 in the multiplicative form gives the bound

$$\left| \frac{\mathcal{A}}{\mathcal{A}} \right| \leq \frac{|\mathcal{A}\mathcal{A}|^2}{|\mathcal{A}|}$$

Hence,

$$|\mathcal{A}\mathcal{A}| \geq c_1 (N \log p)^{3/4} \tag{6}$$

for some absolute constant $c_1 > 0$.

Denote $I = \{1, 2, \dots, N\}$. Let J be the number of solutions of the congruence

$$(n_1 + m_1)!(n_2 + m_2)!(n_3 + m_3)!xy \equiv \lambda \pmod{p},$$

in variables n_1, n_2, m_1, m_2, x, y satisfying

$$n_1, n_2, m_1, m_2 \in I, \quad x, y \in \mathcal{A}.$$

To prove Theorem 2 it suffices to show that there is a constant $C > 0$ such that $J > 0$ for $N = \lfloor Cp^{11/12} \rfloor$. We express J via character sums and get

$$J = \frac{1}{p-1} \sum_{\chi} \sum_{n_1, n_2, m_1, m_2 \in I} \sum_{x, y \in \mathcal{AA}} \chi((n_1 + m_1)!(n_2 + m_2)!(n_3 + m_3)!xy) \chi(\lambda^{-1}).$$

Separating the term that corresponds to the principal character $\chi = \chi_0$ and following the standard argument we obtain

$$J \geq \frac{N^6 |\mathcal{AA}|^2}{p-1} - \frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{n, m \in I} \chi((n+m)!) \right|^3 \left| \sum_{x \in \mathcal{AA}} \chi(x) \right|^2.$$

Applying Lemma 3 and the identity

$$\frac{1}{p-1} \sum_{\chi} \left| \sum_{x \in \mathcal{AA}} \chi(x) \right|^2 = |\mathcal{AA}|,$$

gives

$$J \geq \frac{N^6 |\mathcal{AA}|^2}{p-1} - c_2 N^{21/4} p^{3/8} |\mathcal{AA}|,$$

where $c_2 > 0$ is an absolute constant. Using (6) we obtain

$$\begin{aligned} J &\geq \frac{N^{21/4} |\mathcal{AA}|}{p-1} \left(|\mathcal{AA}| N^{3/4} - c_2 p^{11/8} \right) \\ &\geq \frac{N^{21/4} |\mathcal{AA}|}{p-1} \left(c_1 N^{3/2} (\log p)^{3/4} - c_2 p^{11/8} \right). \end{aligned}$$

Hence, taking $N = \lceil 2(c_2/c_1)^{2/3} p^{11/18} / (\log p)^{1/2} \rceil$, we get $J > 0$, which finishes the proof of our theorem.

5 Remarks

In the proof of Theorem 2 we used the fact that for $N < p^{1-\varepsilon}$ one has the bound

$$|\mathcal{A}(0, N) \mathcal{A}(0, N)| \gg (N \log N)^{3/4}.$$

We note that this bound can significantly be improved for small values of N . For example, let $N < p^{1/2}$. For any positive integers $n, m \leq N$ we have

$$\frac{n}{m} \pmod{p} \subset \frac{\mathcal{A}(0, N) \mathcal{A}(0, N)}{\mathcal{A}(0, N) \mathcal{A}(0, N)}.$$

Note that in the range $n, m < p^{1/2}$ for distinct rational numbers n/m correspond distinct residue classes $n/m \pmod{p}$. Therefore,

$$\begin{aligned} \left| \frac{\mathcal{A}(0, N)\mathcal{A}(0, N)}{\mathcal{A}(0, N)\mathcal{A}(0, N)} \right| &\geq \# \left\{ \frac{n}{m}; n, m \in [1, N] \cap \mathbb{Z}, \gcd(n, m) = 1 \right\} \\ &= \left(\frac{6}{\pi^2} + o(1) \right) N^2 \end{aligned}$$

as $N \rightarrow \infty$. Thus, in the range $N < p^{1/2}$ we have $|\mathcal{A}(0, N)\mathcal{A}(0, N)| \gg N$.

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